P.G. Sem - III , Functional Analysis CC - XI

Ques — A linear transformation T from a normed linear space E into a normed linear space F is continuous iff the image T(S) of the closed unit sphere $S = \{x \in E : \|x\| \leq 1\}$ of E is a bounded set in F.

Soln — Let T be continuous. Then there exists $m > 0$ such that

$$\|T(x)\| \leq m \|x\| \quad \forall \; x \in E$$

Now $x \in S \Rightarrow \|x\| \leq 1$

$$\Rightarrow \|T(x)\| \leq \Rightarrow T(x) \in S_m[0]$$

Hence $T(S) \subseteq S_m[0]$, Hence T(S) is bounded set in F.

Conversely, suppose that T(S) is bounded set in F. Then there exists a closed sphere $S_\varepsilon[0]$ such that

$$T(S) \subseteq S_\varepsilon[0]$$

Now if $x = 0$, then clearly

$$\|T(x)\| \leq \varepsilon \|x\|$$

If $x \neq 0$, take $y = \dfrac{x}{\|x\|}$ then

$$\|y\| = 1$$

Hence $y \in S$, Hence $T(y) \in T(S) \subseteq S_\varepsilon[0]$

Therefore, $\|T(y)\| \leq \varepsilon$ ie $\left\|T\left(\frac{x}{\|x\|}\right)\right\| \leq \varepsilon$

ie $\|T(x)\| \leq \varepsilon \|x\|$

Hence by T is continuous.